EXTENSIONS AND LOW DIMENSIONAL COHOMOLOGY THEORY OF LOCALLY COMPACT GROUPS. II(1)

BY
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Introduction. In a previous paper [8], we have defined a sequence of cohomology groups $H^r(G,A)$ defined when G is a locally compact group and A is an abelian locally compact group on which G acts continuously as a group of automorphisms. The structure of these cohomology groups, particularly the two dimensional groups, was discussed in this paper; here we shall discuss the structure of these cohomology low dimensional groups under more general assumptions on G. Explicitly we shall be concerned with the class of locally compact groups G which have the property that G/G_0 is compact (where G_0 is the connected component of the identity in G). For simplicity we shall call such groups almost connected. When G is in fact connected, then we can obtain more explicit information by use of a principal series of normal subgroups in G discussed by Iwasawa [3], in conjunction with a spectral sequence. Also we shall introduce the notion of splitting groups and use them to discuss the problem of topologizing the two dimensional group $H^2(G,A)$ of extensions of G by A. The notation and terminology used will be that of [8].

Chapter I

1. We shall begin with some results on the structure of the cohomology groups of G when the coefficient module is a finite dimensional vector space.

Proposition 1.1. *Let G be locally compact and H a normal subgroup. Suppose that $H^r(K,V)$ is a finite dimensional vector space for $r=1,2$ and $K=H$ or G/H for any finite dimensional K-module V. Then the same is true for $K=G$.*

Proof. The algebraic part of this proof is trivial. If V is a given G-module, we consider the spectral sequence E_r^{ji} defined by H which abuts to $H^*(G,V)$. It suffices to prove that the $E_2^{i,2-i}$ are finite dimensional, and this will follow from the hypothesis if Theorem 1.1 of [8] is applicable, and if $H^1(H,V)$ has the proper topology. Explicitly, $B^1(H,V)$ is finite dimensional since V is, and $H^1(H,V)$

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is finite dimensional by assumption; thus so is $Z^1(H,V)$. As this last group consists of continuous cocycles, the compact open topology is compatible with the vector space structure. This topology is then the usual one on a finite dimensional vector space, and $B^1(H,V)$ is closed. Theorem 1.1 of [8] applies and E_2^{11} is isomorphic to $H^1(G/H,H^1(H,V))$ which is finite dimensional. As the other summands, E_2^{02} and E_2^{02}, are clearly finite dimensional, the proof is complete.

This result can be used inductively to prove the following result which will be of great use in the sequel.

THEOREM 1.1. *If G is locally compact and almost connected, then $H^r(G,V)$ is finite dimensional and $H^r(G,A)=0$ if G is compact for $r=1,2$ and any finite dimensional G-module V.*

Proof. If G_0 denotes the connected component of G, then [3, §5] there is a series $G \supset G_0 \supset G_1 \supset G_2 \supset \cdots \supset G_n = \{0\}$ of normal subgroups of G_0 so that G_0/G_1 is a connected semi-simple Lie group (with no compact components) and so that the quotients G_i/G_{i+1} for $i \geqq 1$ are either compact groups or vector groups. Finally G/G_0 is compact (totally disconnected) by hypothesis. For a connected semi-simple Lie group H it is well known (or can be easily proved in terms of Borel cochains) that $H^1(H,V)=(0)$, and $H^2(H,V) \simeq \mathrm{Hom}(\pi_1(H),(V)_H)$ (cf. the remarks following Theorem A and Lemma 1.1 of [8]). Since the fundamental group $\pi_1(H)$ is finitely generated, both of these cohomology groups are finite dimensional. For a vector group H, $H^r(H,V) \simeq H^r(h,V)$ for $r=1,2$ where h is the Lie algebra of H (cf. remarks following Theorem A of [8]) and so these two cohomology groups are finite dimensional. Finally, for a compact group H, $H^r(H,V)=(0)$ for $r=1,2$ by Theorem 2.3 of [8]. Proposition 1.1 applied at each step of this normal series yields the desired results.

REMARKS. (1) The conclusions of the previous theorem may be false if G is not almost connected. Let $G = R + \sum_{i=1}^{\infty} Z$; if G is given the trivial representation on a one dimensional vector space, then $H^1(G,R) \simeq R + \prod_{i=1}^{\infty} R$, and $H^2(G,R)$ contains a vector subspace of the form $\prod_{i=1}^{\infty} R$. Thus both of these cohomology groups are of (uncountably) infinite dimension.

(2) The proof of Proposition 1.1 also shows the following: if H is a compact normal subgroup of G, then the inflation homomorphism of $H^2(G/H,(V)^H)$ into $H^2(G,V)$ is an isomorphism onto. This observation could be used to give a slightly different proof of Theorem 1.1.

2. We shall now show how the results of the previous section can be used to study the structure of the much more interesting groups $H^2(G,A)$ where A is a toroidal group. If T^n is a G-module then we have an exact sequence

$$0 \to Z \xrightarrow{i} R^n \xrightarrow{\pi} T^n \to 0.$$

Since R^n is the universal covering of T^n, this sequence becomes in the obvious fashion a G sequence. The following sequence of cohomology is then exact

$$(*) \qquad \rightarrow H^2(G,Z^n) \rightarrow H^2(G,R^n) \xrightarrow{\pi_2} H^2(G,T^n) \rightarrow H^3(G,Z^n).$$

If G is almost connected, then $H^2(G,R^n)$ is a finite dimensional vector space. The results of this section state that both the kernel and cokernel of the map π_2 are denumerable. The proof of these statements about π_2 will proceed in a series of several steps.

PROPOSITION 1.2. *If G is a Lie group with a finite number of comp-ponents and A a discrete (countable) G-module, then $H^1(G,A)$ and $H^2(G,A)$ are denumerable. If A is the direct limit* $\lim d_\alpha A_\alpha$ *of discrete G-modules, then $H^i(G,A) \simeq \lim d_\alpha H^i(G,A_\alpha)$ for $i = 1,2$.*

Proof. If G_0 is the connected component of the identity in G, we have a spectral sequence E_r^{ji} defined by G_0 which ends in $H^*(G,A)$. Now the terms E_2^{10} and E_2^{20} are denumerable since G/G_0 is finite and $A = A^{G_0}$ is denumerable. Since $Z^1(G_0,A) = (0)$ by reasons of continuity, it follows from Theorem 1.1 of [8] that $E_2^{j1} = (0)$ for all j. In order then to establish the first result it suffices to prove that $E_2^{02} \simeq H^2(G_0,A)^{G/G_0}$ is denumerable; that is, we may and so assume that $G = G_0$. It is then well known that $H^2(G,A) \simeq \mathrm{Hom}(\pi_1(G),A)$ where $\pi_1(G)$ is the fundamental group of G. Since $\pi_1(G)$ is finitely generated, the result follows immediately.

The second assertion for dimension one is trivial. In dimension two, if $G = G_0$, the isomorphism above of $H^2(G,A_\alpha)$ with $\mathrm{Hom}(\pi_1(G),A_\alpha)$ yields the result immediately in virtue of the fact that $\pi_1(G)$ is finitely generated. In general we consider a family of spectral sequences ${}^\alpha E_r^{ji}$ similar to E_r^{ji} but abutting to the module $H^*(G,A_\alpha)$. In addition to maps of $H^*(G,A_\alpha)$ into $H^*(G,A)$ there are maps from ${}^\alpha E_r^{ji}$ into E_r^{ji} and these maps are compatible with the limiting process. In order to prove that the resulting map of $\lim d_\alpha H^2(G,A_\alpha)$ into $H^2(G,A)$ is an isomorphism, a trivial algebraic verification shows that it is sufficient to prove that the resulting maps from $\lim d_\alpha {}^\alpha E_2^{ji}$ into E_2^{ji} are isomorphisms for all j and $i = 0,1,2$. (Actually only six of these isomorphisms are necessary.) From the remarks following Theorem 1.1 of [8], it is clear that $E_2^{ji} \simeq H^j(G/G_0, H^i(G_0,A))$ and similarly for ${}^\alpha E_2^{ji}$ for each of the foregoing pairs of indices. (We view $H^i(G_0,A)$ as a discrete group.) However, we have shown that $\lim d_\alpha H^i(G_0,A_\alpha)$ is isomorphic to $H^i(G_0,A)$ for $i = 0,1,2$. Since formation of the cohomology is the finite group G/G_0 commutes with direct limits, the necessary isomorphisms of $\lim d_\alpha {}^\alpha E_2^{ji}$ with E_2^{ji} are established.

COROLLARY. *In the above notation, if the A_α are sub-G-modules of A, then the natural map of* $\lim d_\alpha H^3(G,A_\alpha)$ *into $H^3(G,A)$ is injective.*

Proof. In this case we may write an exact sequence

$$\to H^2(G, A/A_\alpha) \to H^3(G, A_\alpha) \to H^3(G, A).$$

It is known that this sequence remains exact upon taking direct limits; that is,

$$\to \lim d_\alpha H^2(G, A/A_\alpha) \to \lim d_\alpha H^3(G, A_\alpha) \to H^3(G, A)$$

is exact. Since $\lim d_\alpha A/A_\alpha = (0)$ it follows from the proposition that the first group in this sequence is zero; whence the corollary.

PROPOSITION 1.3. *If G is locally compact and almost connected and A is as above, then $H^1(G,A)$ and $H^2(G,A)$ are denumerable.*

Proof. It is known [7, p. 175] that there is a compact normal subgroup H of G so that G/H is a Lie group with a finite number of components. We make use again of the spectral sequence E_r^{ji} defined by H and which abuts to $H^*(G,A)$. The term E_2^{02} is denumerable by Theorem 2.3 of [8], and E_2^{20} is also denumerable by the preceding proposition. Finally since $H^1(H,A)$ is denumerable [8, Theorem 2.3] as is A/A^H, $Z^1(H,A)$ is also denumerable. The compact open topology on this group is discrete so that Theorem 1.1 of [8] applies, and E_2^{j1} is isomorphic to $H^j(G/H, H^1(H,A))$. By the preceding proposition the term E_2^{11} is denumerable. The desired result is then immediate.

Returning to exact sequence (*) at the beginning of this section, we see that $H^2(G,Z^n)$ is countable. Thus we deduce the following result.

COROLLARY. *If G is almost connected, the kernel of the map π_2 of $H^2(G,R^n)$ into $H^2(G,T^n)$ is countable. Also the map π_1 of $H^1(G,R^n)$ into $H^1(G,T^n)$ has countable kernel and cokernel.*

We note in passing that these arguments show that $H^1(G,A)$ is a torsion group; of course these facts in dimension one do not require the elaborate machinery used.

We have also a limit theorem in this context analogous to those of [8].

PROPOSITION 1.4. *If G is almost connected and $G = \lim p_\alpha G_\alpha$ where $G_\alpha \simeq G/H_\alpha$ and H_α are compact and normal, and if $A \simeq \lim d_\alpha A_\alpha$ of sub-G-modules A_α, then*

$$\lim d_\alpha H^2(G_\alpha, A_\alpha^{H_\alpha}) \simeq H^2(G,A).$$

Proof. Let us fix a compact normal subgroup H so that G/H is a Lie group. We may and shall suppose that $H_\alpha \subset H$ for all α since this is true for a cofinal set of indices α. There are spectral sequences ${}^\alpha E_r^{ji}$ (E_r^{ji}) defined by the group extension G/H by H/H_α (G/H by H) which abut to $H^*(G_\alpha, A_\alpha^{H_\alpha})(H^*(G,A))$. It is evident that we may form the limit $\lim d_\alpha {}^\alpha E_r^{ji}$ and that there is map ϕ_r^{ji} of this limit into E_r^{ji}. It may be verified that in order to prove the assertion of the proposition it is sufficient to prove that ϕ_2^{ji} is an isomorphism in bi-degrees (2,0), (0,1), (1,1), (2,1), (0,2) and that ϕ_2^{30} is an injection. It is known that ${}^{\alpha}E_2^{j0} \simeq H^j(G/H, A_\alpha^H)$ and similarly for E_2^{j0}. It is clear that $A^H = \lim d_\alpha A_\alpha^H$ since H is compact. As the A_α^H are sub-G/H-modules, the desiderata for $j = 2,3$ follow from the previous proposition and its corollary. Using the fact that H is compact and A is discrete, we verify that hypothesis (a) of [8, Theorem 1.1] is satisfied for the special sequences ${}^{\alpha}E_r^{ji}, E_r^{ji}$. We have isomorphisms ${}^{\alpha}E_2^{j1} \simeq H^j(G/H, H^1(H/H_\alpha, A_\alpha^{H_\alpha}))$ where $H^1(H/H_\alpha, A_\alpha^{H_\alpha})$ is assigned the discrete topology (similarly for E_r^{ji}). The desired facts for $i = 1$ follow immediately from Theorem 2.3 of [8] and the preceding proposition. Finally, ${}^{\alpha}E_2^{0i} \simeq ({}^{\alpha}E_1^{0i})^{G/H} = H^i(H/H_\alpha, A_\alpha^{H_\alpha})^{G/H}$, and similarly for E_2^{0i}. By Theorem 2.3 of [8], $\lim d_\alpha {}^{\alpha}E_1^{0i} \simeq E_1^{0i}$ for $i = 1,2$. Now the connected component G_0 of G operates trivially on both H and A so that G_0 operates trivially on E_1^{0i} and ${}^{\alpha}E_1^{0i}$. Thus these groups are $G/H \cdot G_0$-modules, but $G/H \cdot G_0$ is a finite group. Since the operation of direct limits commutes with taking elements invariant by a finite group of automorphisms, it follows that $\lim d_\alpha {}^{\alpha}E_2^{0i} \simeq E_2^{0i}$ for $i = 1,2$. This completes the proof.

3. We return now to the question of the cokernel of the map π_2 defined in the last section. We shall need the following technical lemma which is surely known.

Lemma 1.1. *If H is a finitely generated abelian group and A is a discrete (denumerable) H-module, then $H^*(H, A)$ is denumerable.*

Proof. The proof proceeds by induction on the rank of H; we omit the details.

Proposition 1.5. *If G is a connected Lie group, then the cokernel of π_2 is denumerable.*

Proof. Clearly G must operate trivially on the module T^n and hence on R^n. In the special case when G is also simply connected, π_2 is an isomorphism onto by the remarks following Theorem A of [8]. In general let $\tilde{G}$ denote the universal covering group of G and consider the following diagram

$$\begin{array}{ccccc} \xrightarrow{\pi_2} & H^2(G, T^n) & \xrightarrow{\partial} & H^3(G, Z^n) & \longrightarrow \\ & \downarrow i_1 & & \downarrow i_2 & \\ \xrightarrow{\tilde{\pi}_2} & H^2(\tilde{G}, T^n) & \xrightarrow{\tilde{\partial}} & H^3(\tilde{G}, Z^n) & \longrightarrow \end{array}$$

where i_1 and i_2 are the appropriate inflation homomorphisms. By exactness, the map $\tilde{\partial}$ is the zero map and by commutativity the range of ∂ is contained in the kernel of i_2. Thus it will suffice to prove that the kernel of i_2 is denumerable. But this kernel may be estimated by use of a spectral sequence; namely, the one defined by the extension of G by $\pi_1(G)$ and which abuts to $H^*(\tilde{G}, Z^n)$.

Theorem 1.1 of [8] is applicable to this spectral sequence since

$$Z^1(\pi_1(G), Z^n) = H^1(\pi_1(G), Z^n)$$

is countable and discrete in the compact open topology. It follows that the term E_2^{11} is zero since G is connected and Borel homomorphisms are continuous. Therefore the differentiation map d_2^{11} of E_2^{11} into E_2^{30} is trivial and E_3^{30} is canonically isomorphic to $E_2^{30} \simeq H^3(G,Z^n)$. The kernel of the map i_2 as a subgroup of E_3^{30} is then the image of the map d_3^{02} from E_3^{02} into E_3^{30}. Now E_3^{02} is a subgroup of E_2^{02}, and the latter group is isomorphic to $H^2(\pi_1(G),Z^n)$ which is denumerable by Lemma 1.1 since $\pi_1(G)$ is finitely generated. It follows that the kernel of i_2 is denumerable; this completes the proof.

We remark that it also may be proved by refining these techniques that the kernel of the inflation map from $H^4(G,A)$ into $H^4(\tilde{G},A)$ is denumerable if A is any discrete G-module.

PROPOSITION 1.6. *If G is a Lie group with a finite number of components and T^n is a G-module, then the cokernel of π_2 is denumerable.*

Proof. Let G_0 be the connected component of the identity in G. Since G_0 operates trivially on T^n, T^n (and R^n, Z^n) become G/G_0-modules. In the following diagram it must be proved that the image of ∂ is countable:

$$\begin{array}{ccccccc} \longrightarrow & H^2(G,T^n) & \xrightarrow{\partial} & H^3(G,Z^n) & \longrightarrow \\ & \downarrow r_2 & & \downarrow r_3 & \\ \longrightarrow & H^2(G_0,T^n) & \xrightarrow{\partial_0} & H^3(G_0,Z^n) & \longrightarrow . \end{array}$$

Since the range of ∂_0 is known to be countable, it is easily seen to be sufficient to prove that the kernel of the restriction map r_3 is countable. The spectral sequence of the group extension of G/G_0 by G_0 which abuts to $H^*(G_0,Z^n)$ can be used to estimate the size of this kernel K_1. Indeed, it is enough to prove that each of the terms E_2^{12}, E_2^{21}, E_2^{30} of this spectral sequence is countable. Now the groups $H^i(G_0,Z^n)$ are countable (in fact finitely generated) for $i = 0,1,2$ and it follows from the second remark following Theorem 1.1 of [8] that E_2^{ji} is isomorphic to $H^j(G/G_0,H^i(G_0,Z^n))$ for $i = 0,1,2$, (for all i if one allows noncountable discrete modules). Since G/G_0 is finite, all the groups E_2^{ji} for $i = 0,1,2$ are countable (in fact even finite) and this completes the proof.

The final results can now be stated.

THEOREM 1.2. *If G is locally compact and almost connected and T^n is as above, then the cokernel of the map π_2 is countable.*

THEOREM 1.3. *If G and T^n are as above and $G \simeq \lim p_\alpha G_\alpha$ where $G_\alpha = G/H_\alpha$ with H_α compact, then the map*

$$\lim d_\alpha H^r(G_\alpha, T^{n H_\alpha}) \to H^r(G, T^n)$$

is an isomorphism for $r = 2$ and an injection for $r = 3$.

Proof of Theorem 1.2. As before we select a normal compact subgroup H so that G/H is a Lie group. In the diagram

$$\begin{array}{ccc} H^2(G/H,(R^n)^H) & \xrightarrow{\pi_2'} & H^2(G/H,(T^n)^H) \\ \downarrow i_1 & & \downarrow i_2 \\ H^2(G,R^n) & \xrightarrow{\pi_2} & H^2(G,T^n) \end{array}$$

the map π_2' has denumerable cokernel.

If the map i_2 also has denumerable cokernel, then so clearly does π_2 which is the result desired. We consider the spectral sequence of the group extension G of G/H by H abutting to $H^*(G,T^n)$. If the terms E_2^{02}, which is isomorphic to $H^2(H,T^n)^G$, and E_2^{11}, which is isomorphic to $H^1(G/H,H^1(H,T^n))$ (Theorem 1.1 of [8]), are denumerable, then the kernel of i_2 is denumerable. However, both of these terms are clearly denumerable by the corollary of Theorem 2.2 of [8] and Proposition 1.2. This completes the proof.

Proof of Theorem 1.3. For $r = 2$, the proof is carried out exactly as in the proofs of Theorem 2.2 of [8] and Proposition 1.4. For $r = 3$, the proof is carried out exactly as in Theorem 2.2 of [8]. We shall omit the details.

Chapter II

1. We consider now the question of topologizing the group of extensions $H^2(G,A)$ of G by a G-module of A. If G is almost connected, the results of the preceding chapter show how to do this. If A is vector group, then $H^2(G,A)$ is a finite dimensional vector space and the ordinary topology is unique if one imposes the natural requirement that $H^2(G,A)$ is a topological vector space. If A is discrete, $H^2(G,A)$ is countable and the discrete topology is unique subject to the natural requirement in this case that $H^2(G,A)$ be a locally compact group. We are then essentially reduced to the case of a compact group A; we must it seems for this treatment restrict to the case when A is a finite dimensional torus. In this case let V be the universal covering of A, and we use the results of the last chapter concerning the G homomorphism π of V onto A to examine more closely the structure of $H^2(G,A)$.

Let D_1' be the image of $H^2(G,Z^n)$ in $H^2(G,V)$, W the subspace generated by D_1'. We may find a linear map $\alpha \to a(\alpha)$ of $H^2(G,V)$ into $Z^2(G,V)$ so that $a(\alpha)$ is a representative of α and so that $a(\alpha)$ takes values in $Z^n \subset V$ when α runs through a basis of W chosen from D_1'. If D_1'' denotes the subgroup of $H^2(G,V)$ consisting of those α so that $a(\alpha)$ takes values in Z^n, then it is clear that there is a neighborhood U of 0 in $H^2(G,V)$ so that $U \cap D_1'' = (0)$. The subgroup D_1'' is then closed, finitely generated and contains a basis of W. Let K_0 denote the abstract Lie group $H^2(G,V)/D_1''$. Then the projection $\pi(a(\alpha))$ of $a(\alpha)$ into $H^2(G,A)$ defines an isomorphism $\beta' \to b(\beta')$, $\beta' \in K_0$, of K_0 into $Z^2(G,A)$. We need now the following simple fact.

LEMMA 2.1. *The groups $B^r(G,A)$ are divisible for all r.*

Proof. We may select for any positive integer n a Borel function ϕ_n of A into itself so that $n\phi_n(a) = a$ for all $a \in A$. If $f \in C^r(G,A)$ then $\phi_n f$ also belongs to $C^r(G,A)$ and $n(\phi_n f) = f$. Thus $C^r(G,A)$ is divisible and $B^r(G,A)$, $r > 0$, being a quotient group of $C^{r-1}(G,A)$, is also divisible.

Returning to the above, we see that the group I generated by the two divisible subgroups $B^2(G,A)$ and $b(K_0)$ of $Z^2(G,A)$ is divisible. Theorem 1.2 then asserts that I is of countable index in $Z^2(G,A)$. The subgroup I is a direct summand and we may choose a homomorphism $\beta'' \to b(\beta'')$ of the group $D_2 = Z^2(G,A)/I = \operatorname{coker}(\pi_2)$ into $Z^2(G,A)$ implementing this direct sum. Finally, let K denote the abelian Lie group $K_0 + D_2$. If $\beta = (\beta',\beta'')$ is in K where $\beta' \in K_0$, $\beta'' \in D_2$, we define a cocycle $b(\beta) = b(\beta') + b(\beta'')$ in $Z^2(G,A)$. The correspondence $\beta \to b(\beta)$ is additive, and we have in addition the following facts.

PROPOSITION 2.1. *Every cocycle in $Z^2(G,A)$ is equivalent to some $b(\beta), \beta \in K$; $b(\beta) \sim 0$ iff $\beta = (\beta',0)$ where β' belongs to the countable subgroup $D_1'/D_2'' = D_1$ of the maximal compact subgroup of K_0.*

Thus the group $H^2(G,A)$ is represented as a quotient of an abelian Lie group K by a possibly nonclosed denumerable subgroup. As such it inherits a topology—that of K/D_1. The situation is quite analogous to the representation of the dual (equivalence classes of irreducible representations) of a locally compact group G as the quotient of a standard Borel space by a possibly nonmeasurable equivalence relation [5]. Of course with the group of extensions $H^2(G,A)$, we must show that the representation above is intrinsic in a suitable sense and independent of the rather *ad hoc* manner in which we obtained it. This we propose to do.

2. We continue the notation established above. If (s,t) is a fixed element of $G \times G$, we defined $[\phi(s,t)]\beta = [b(\beta)](s,t)$, $\beta \in K$. Then $\phi(s,t)$ is a map from K to A.

THEOREM 2.1. *For each (s,t) $G \times G$, $\phi(s,t)$ belongs to* Hom(K,A) *(continuous homomorphisms). If* Hom$(K,A) = B$ *is given the compact open topology (in which it is locally compact), and if it is given the structure of a G-module by $(sf)\beta = s(f(\beta))(s \in G,\ f \in B)$, then ϕ belongs to $Z^2(G,B)$.*

Proof. We first observe that $\phi(s,t)$ is an algebraic homomorphism from K to A. Let $\phi_1(s,t)$ and $\phi_2(s,t)$ be the restrictions of $\phi(s,t)$ to K_0 and D_2 respectively. We define $[\psi(s,t)]\alpha = [a(\alpha)](s,t)$ for $\alpha \in H^2(G,V)$. Then $\psi(s,t)$ is a linear map of $H^2(G,V)$ into V and is hence continuous in the usual topologies on these vector spaces. It is clear that $[\phi_1(s,t)](p(\alpha)) = [\pi(\psi(s,t))](\alpha)$ where p is the projection of $H^2(G,V)$ onto K_0. This formula together with the fact that $\psi(s,t)$ is continuous shows that $\phi_1(s,t)$ is continuous. It is clear that the correspondence between $f\in \mathrm{Hom}(K_0,A)$ and those g in $\mathrm{Hom}(H^2(G,V),V)$ which map D_1'' into Z^n given by $f(p(\alpha)) = \pi(g(\alpha))$ ($a \in H^2(G,V)$, p and π as above) is biunique. Moreover the correspondence is a homeomorphism in the compact open topologies. Therefore, in order to prove that ϕ_1 is a Borel function on $G \times G$ to $\mathrm{Hom}(K_0,A)$, it suffices to prove that ψ is a Borel function on $G \times G$. In virtue of the definition of ψ above, this is equivalent to the fact that $[a(\alpha)](s,t)$ is a Borel function on $G \times G$ to V for each fixed α. But this condition is *a priori* verified since $a(\alpha)\in Z^2(G,V)$.

The map $\phi_2(s,t)$ is a continuous map from D_2 into A as D_2 is discrete. Then in order to prove that ϕ is a Borel map from $G \times G$ to $\mathrm{Hom}(K,A)$ it suffices to show that ϕ_2 is a Borel map from $G \times G$ into $\mathrm{Hom}(D_2,A)$. If $\gamma_0\in \mathrm{Hom}(D_2,K)$, and $\beta_i, i = 1,\cdots,n$, is a finite set in D_2, and U is a neighborhood of 0 in A, then the sets $\{\gamma:(\gamma-\gamma_0)\beta_i\in U,\ i=1,\cdots,n\}$ form a basis for the topology in $\mathrm{Hom}(D_2,A)$. Since the topology is separable metric, ϕ_2 is a Borel function if the counterimage of each of these sets is a Borel set. But

$$\phi_2^{-1}(\cdot) = \bigcap_{i=1}^{n} \{(s,t):[b(\beta_i)](s,t) - \gamma_0(\beta_i)\in U\}.$$

This set is a Borel set since each $b(\beta_i)$ is a Borel function on $G \times G$. This completes the proof but to observe that it is trivial that ϕ satisfies the cocycle relation and hence is in $Z^2(G,B)$.

The element ϕ in $Z^2(G,B)$ then defines a locally compact extension H of G by the G-module B. (More properly the class of ϕ defines an equivalence class of extensions of G by B.) The G-module A becomes in the obvious fashion an H-module.

PROPOSITION 2.2. *In the restriction inflation sequence,*

$$H^1(H,A) \xrightarrow{\text{res}} H^1(B,A)^G \xrightarrow{\text{tg}} H^2(G,A) \xrightarrow{\text{inf}} H^2(H,A),$$

the transgression map tg *is surjective.*

Proof. By duality, K is mapped topologically onto a closed subgroup K^* of $H^1(B,A) = \mathrm{Hom}(B,A)$ by $\beta \to f(\beta)$, where $[f(\beta)](\gamma) = \gamma(\beta)$, $\gamma\in B$. ($H^1(B,A)$ is given the compact open topology, as usual.) The subgroup K^* is left invariant by G for $[sf(\beta)](\gamma) = s([f(\beta)](s^{-1}\gamma)) = s[(s^{-1}\gamma)(\beta)] = \gamma(\beta) = [f(\beta)](\gamma)$, $\gamma\in B$, $s\in G$. From [8, Chapter 1, §5], an explicit cocycle representative of $\mathrm{tg}[f(\beta)]$ is the cocycle on G with values in A given by $f(\beta)[\phi(s,t)] = [\phi(s,t)](\beta) = [b(\beta)](s,t)$, $s,t\in G$. Thus by Proposition 2.1, tg, even restricted to K^*, is surjective.

The kernel of the transgression map is some (possibly nonclosed) subgroup P of $H^1(B,A)$. Its intersection with the subgroup K^* is the subgroup D_1^* corresponding to the denumerable subgroup D_1 of K(Proposition 2. 1). Thus $H^2(G,A)$ inherits a locally compact (not necessarily Hausdorff) topology, that of $H^1(B,A)^G/P$. It is not immediately clear at present that the topology for $H^2(G,A)$ mentioned previously (end of §1)—that of K/D_1—is the same. This is in fact so.

PROPOSITION 2.3. *The natural map of* K^*/D_1^* *onto* $H^1(B,A)^G/P$ *is a homeomorphism.*

We shall defer the proof of this until after Theorem 2.2.

3. Proposition 2.2 may be alternately phrased by saying that the inflation map from $H^2(G,A)$ to $H^2(H,A)$ is the zero map. That is, every extension of G by A is split when inflated to H. This leads us to the following definition.

DEFINITION 2.1. If G is any locally compact group and A any locally compact G-module, a locally compact extension of G by a group B, H is a splitting group for G and A if the inflation map of $H^2(G,A)$ into $H^2(H,A)$ is the zero map.

If H is a splitting group for G and A, then $H^1(B,A) = \mathrm{Hom}(B,A)$ is a complete and separable metric group in the compact open topology. Thus $H^2(G,A)$, which is isomorphic to $H^1(B,A)^G/\ker(\mathrm{tg})$ receives a topology which we call the H-topology.

We have shown that if G is almost connected and A is a toroidal G-module, then there is a splitting group H. The same method as was used in §1 can clearly also be used to show that there is a splitting group when G is almost connected and when A is a finite dimensional vector group. We have remarked that if A is toroidal and H is the splitting group introduced above, then the H topology on $H^2(G,A)$ is, in the terminology of §1, just the topology induced on $H^2(G,A)$ by abstract isomorphism between it and K/D_1. Of course many arbitrary choices went into the construction of H; also, there may be other totally different splitting groups giving different H topologies. The following theorem however shows that these considerations are irrelevant and that the resulting topology is in a sense intrinsic.

THEOREM 2.2. *Let G be locally compact and let A be a toroidal G-module. If G and A have two splitting groups H_1 and H_2, then the H_1 and H_2 topologies on $H^2(G,A)$ are identical.*

The proof proceeds in a series of steps. Let H_i be an extension of G by B_i. Since $\mathrm{Hom}(B_i,A)$ is canonically isomorphic to $\mathrm{Hom}(B_i/\overline{[B_i,B_i]},A)$, we may and shall factor H_i by the normal subgroup $\overline{[B_i,B_i]}$ thereby changing nothing of significance. Thus H_i is now an extension of G by an abelian group B_i. Let ϕ_i be a cocycle in $Z^2(G,B_i)$ describing this extension. We form the cocycle $\phi = \phi_1 + \phi_2$ of G with coefficients in $B = B_1 + B_2$. It defines an abelian extension of G by B which we call H. If we represent H as triples $(b_1,b_2,s) b_i \in B_i$, $s \in G$ and H_i as pairs (b_i,s), following the cocycles ϕ, ϕ_i, there are homomorphisms of H onto H_i given by $(b_1,b_2,s) \to (b_i,s)$. These maps are evidently Borel maps, hence continuous.

We then have a commuting diagram:

$$
\begin{array}{ccccc}
H^1(H_1,A) & \longrightarrow & H^1(B_1,A)^G & \xrightarrow{\mathrm{tg}_1} & H^2(G,\ A) \\
\downarrow & & \downarrow f_1 & & \downarrow i_1 \\
H^1(H,A) & \longrightarrow & H^1(B,A)^G & \xrightarrow{\mathrm{tg}} & H^2(G,\ A) \\
\uparrow & & \uparrow f_2 & & \uparrow i_2 \\
H^1(H_2,A) & \longrightarrow & H^1(B_2,A)^G & \xrightarrow{\mathrm{tg}_2} & H^2(G,\ A).
\end{array}
$$

It is clear that $C = H^1(B,A)^G$ is locally compact (it is a closed subgroup of $\sum_{i=1}^n \hat{B}$) and that $C \simeq C_1 + C_2$ where $C_i = H^1(B_i,A)^G$. The maps i_1 and i_2 are identity maps.

We first observe that H is a splitting group since tg is clearly surjective. In order to establish the theorem it will be enough to show that the map i_j is a homeomorphism from the H_j topology to the H topology.

PROPOSITION 2.4. *The maps i_1 and i_2 are continuous.*

Proof. If U is H-open in $H^2(G,A)$, then $\mathrm{tg}^{-1}(U)$ is open in C, and $f_1^{-1}(\mathrm{tg}^{-1}(U))$ is open in C_1. Since tg_1 is an open map of C_1 onto $H^2(G,A)$, $V = \mathrm{tg}_1(f_1^{-1}(\mathrm{tg}^{-1}(U)))$ is an H_1 open set in $H^2(G,A)$. Since tg is surjective when restricted to $f_1(C_1)$, it follows that $i_1^{-1}(U) = V$ is H_1 open. This completes the proof.

The rest of the argument will show how to apply the closed graph theorem to i_1 and i_2. Care must be exercised since the topologies in question, though locally compact, are not Hausdorff. Let T_i, T be respectively the kernels of tg_i and tg in C_i and C, and let $\bar{T}_i$ and $\bar{T}$ denote their closures. We denote by $\tilde{T}$ the subgroup of C generated algebraically by $(\bar{T}_1, e_2)$ and T. (Here we are viewing C as $C_1 + C_2$ with the identifications provided by the maps f_i above; e_i, e denote as usual the respective neutral elements of these groups.) It is clear from the commutative diagram above that

$$\tilde{T} \cap (C_1, e_2) = (\bar{T}_1, e_2).$$

Since $\bar{T}_1$ is a closed subgroup of the locally compact group C_1, there is a Borel set M_1 in C_1 meeting each coset of $\bar{T}_1$ in one point. Let $M = (M_1, e_2)$. If M contains two points t and t' of the same coset of $\tilde{T}$, then the second components of t and t' vanish and their first components differ by an element of $(\bar{T}_1, e_2)$ as a result of the last paragraph. It follows then that $t = t'$. Finally since tg is surjective on (C_1, e_2), every coset of T meets (C_1, e_2), and thus every coset of $\tilde{T}$ must meet M.

PROPOSITION 2.5. *The subgroup $\tilde{T}$ is closed in C.*

Proof. We have shown that the Borel set M is a cross section of $C/\tilde{T}$ in C. Now $Z^1(H,A)$ is a complete separable metric space in the compact open topology, and it is trivial that the composed map $Z^1(H,A) \to H^1(H,A) \to C$ is a continuous map into C. It follows [4, p. 360] that its image T is an analytic subset of C. Then the subset $T \times (\bar{T}_1, e_2)$ of $C \times C$ is an analytic set. Its image under the composition map of $C \times C$ into C is then also analytic. This image is of course just $\tilde{T}$.

Now let N be a Borel subset of M and N' its complement in M. Then the sets $N \times \tilde{T}$ and $N' \times \tilde{T}$ in $C \times C$ are analytic. Their images under the composition map into C are just the saturations of N and N' with respect to the subgroup $\tilde{T}$, let us say E and E'. It follows as before that E and E' are analytic. But E and E' are complements of each other in C, and thus by Corollary 1, p. 395 of [4], E and E' are Borel subsets of C. Now as N runs over a countable separating family of Borel subsets of M, the projections of the corresponding sets E into $C/\tilde{T}$ define a countable separating family for the quotient Borel structure on $C/\tilde{T}$. By a theorem of Mackey [5, Theorem 7.2], $\tilde{T}$ is a closed subgroup of C.

The following corollary is now clear.

Corollary. *The subgroups $\bar{T}$ and $\tilde{T}$ of C are identical.*

We can now prove that i_1 is a homeomorphism. This map may be viewed as a continuous one-to-one map of C_1/T_1 onto C/T. Therefore it must map $\bar{T}_1/T_1$ into $\bar{T}/T$. The corollary above, however, immediately implies that i_1 maps $\bar{T}_1/T_1$ *onto* $\bar{T}/T$. Then i_1 induces a one-to-one continuous map j_1 of $C_1/\bar{T}_1$ onto $C/\bar{T}$. It is clear that j_1 is bicontinuous if and only if i_1 is. But now $C_1/\bar{T}_1$ and $C/\bar{T}$ are locally compact and Hausdorff. Thus the closed graph theorem [4, pp. 399-400] is applicable and it follows that j_1 is bicontinuous. This proves Theorem 2.2.

We have deferred the proof of Proposition 2.3 until now since the method involved is virtually identical to the method used above. We merely sketch the proof. In the terminology of Proposition 2.3, one proves easily that the closure $\bar{P}$ of P is generated algebraically by P and the closure $\bar{D}_1^*$ of D_1^*. The result follows exactly as above.

4. If A is a toroidal G-module and G and A have a splitting group, $H^2(G,A)$ may be given a well-defined topology which we simply term the splitting group topology. With respect to it, $H^2(G,A)$ is a topological group—but not necessarily Hausdorff. If G is almost connected, then we have seen that this topology is exactly the topology that $H^2(G,A)$ receives as the result of the abstract isomorphism between it and K/D_1 which we constructed in §1. The following theorems show that this topology, whenever it exists, behaves as one would expect it to.

Theorem 2.3. *Let A_i be toroidal G_i-modules respectively, $i = 1,2$, and suppose that G_i and A_i have splitting groups. Let f be a continuous homomorphism of G_1 into G_2 and let α be a compatible homomorphism of A_2 into A_1; that is,*

$s[\alpha(a_2)] = \alpha(f(s)[a_2])$ for $\alpha_2 \in A_2$, $s \in G_1$. Then the corresponding map f^* of $H^2(G_2, A_2)$ into $H^2(G_1, A_1)$ is continuous.

Proof. Let H_i be splitting groups for G_i and A_i with abelian kernels B_i and cocycles $\phi_i \in Z^2(G_i, B_i)$. Let $B = B_1 + B_2$, and since B_2 can be viewed as a G_1-module by use of the map f, we may give B the structure of a G_1-module. We now define a 2 cocycle ϕ of G_1 in B by the formula $\phi(s,t) = (\phi_1(s,t), \phi_2(f(s), f(t)))$. Since ϕ clearly belongs to $Z^2(G_1, B)$, it defines an extension H of G_1 by the G_1 kernel B. It is clear, just as in the proof of Theorem 2.2, that H is a splitting group for G_1 and A_1.

Now H may be viewed as triples (b_1, b_2, s_1) and H_2 as pairs (b_2, s_2) $(b_i \in B_i$, $s_i \in G_i)$. The map h which sends (b_1, b_2, s) into $(b_2, f(s))$ is a Borel homomorphism of H onto H_2 and hence continuous. It projects B onto B_2 and induces f on $G_1 = H/B$ to $G_2 = H_2/B_2$. We are then led to a diagram of the form

$$\begin{array}{ccccc} H^1(H, A_1) & \longrightarrow & H^1(B, A_1)^{G_1} & \xrightarrow{\text{tg}} & H^2(G_1, A_1) \\ \uparrow & & \uparrow & & \uparrow f^* \\ H^1(H_2, A_2) & \longrightarrow & H^1(B_2, A_2)^{G_2} & \xrightarrow{\text{tg}_2} & H^2(G_2, A_2). \end{array}$$

Observe that the map h is compatible with the homomorphism α from the H_2-module A_2 into the H-module A_1. Since all maps in question are continuous, and since the topology on $H^2(G_1, A_1)$ is defined by the splitting group H, the argument of Proposition 2.4 may be applied directly to show that f^* is continuous. This completes the proof.

THEOREM 2.4. *Let A be a toroidal G-module, and suppose that $G \simeq G_1/G_2$ where G_1 and G_2 are locally compact. Then the transgression map,* tg, *of $H^1(G_2, A)^G$ into $H^2(G, A)$ is continuous. (The group G_2 is assumed to operate trivially on A and $H^1(G_2, A)$ is given the compact open topology.)*

The proof is entirely similar to those already given, and is omitted.

There remains the question of the continuity of the coboundary homomorphism of $H^1(G, A'')$ into $H^2(G, A')$ deduced from a short exact sequence $0 \to A' \to A \to A'' \to 0$. Concerning this question we can say only the following. If G is almost connected, and if A', A and A'' are toroidal, the splitting group constructed by means of Theorem 2.1 shows immediately that this coboundary homomorphism is continuous when $H_1(G, A'')$ is given the quotient topology inherited from the compact open topology on $Z^1(G, A'')$. We omit the details.

It is entirely possible that an H topology on $H^2(G, A)$ defined by a splitting group H for G and A may be rather trivial. For example, the closure of the neutral element L may be a large subgroup or even the whole group. In case A is a toroidal group where we have a unique topology, L is a uniquely defined subgroup. In analogy with the dual of a locally compact group [5] we say that $H^2(G,A)$ is smooth whenever $L = 0$. If G is almost connected, the explicit construction of a splitting group (Theorem 2.2) gives a necessary and sufficient condition for smoothness. Let $0 \to Z^n \to V \to A \to 0$ be the exact G sequence associated with universal covering V of A.

PROPOSITION 2.6. *In the notation above $H^2(G,A)$ is smooth if and only if the image of $H^2(G,Z^h)$ in $H^2(G,V)$ is closed.*

The following theorem throws some light on the nature of the subgroup L above and on the nature of splitting group topologies in general.

THEOREM 2.5. *Let G be locally compact and let A be an arbitrary locally compact G-module. Let H be a splitting group for G and A. If $\alpha_n \to 0$ in $H^2(G,A)$ in the H topology, then there is a sequence a_n of cocycles with $a_n \in \alpha_n$ so that $a_n \to 0$ uniformly on compact subsets of $G \times G$.*

Proof. We may assume that H is an extension of G by an abelian kernel B. We may choose a cocycle ϕ of G in the G-module B describing this extension so that ϕ is bounded on bounded sets in $G \times G$. Then if $\alpha_n \to 0$ in $H^2(G,A)$, we may find a sequence γ_n in $H^1(B,A)^G$ which tends to 0 and so that $\mathrm{tg}\gamma_n = \alpha_n$. But now $a_n(s,t) = \gamma_n(\phi(s,t))\,(s,t \in G)$, is a cocyle representative of α_n. Since ϕ is bounded on bounded sets in $G \times G$ and $\gamma_n \to 0$ uniformly on compact subsets of B, we have the desired result.

5. We shall conclude this chapter with some remarks concerning the connection of the foregoing with the subject of projective representations [6] of a group G. We consider the circle group T as a trivial G-module for a fixed locally compact group G and let H be a splitting group for G and T (hereafter referred to as a splitting group for G). If H is an extension of G by B, we have seen that B may be assumed to be abelian, and in this case it is easily seen that B may also be assumed to be central in H.

It follows from the theory of [6] that there is a map from certain ordinary representations of H to projective representations of G. To be explicit, let L be a primary unitary representation of H on a certain Hilbert space X; then L restricted to the central subgroup B must reduce to a multiple of some character $\gamma \in \hat{B} = H^1(B,T)$. If we consider L as a homomorphism of G into the projective unitary group of the Hilbert space X, this map is trivial on B and therefore defines a projective representation M of $G = H/B$. To such a projective representation there is attached [6] a cohomology class $\alpha \in H^2(G,T)$. This class is easily computed, for let $\phi \in Z^2(G,B)$ be a cocycle defining the extension H of G by B, and represent H as pairs (b,s), $b \in B$, $s \in G$ as usual. We define $M'_s = L_{(e,s)}$ for $s \in G$. It is clear that M' is a Borel function to the unitary group in X. Also from the definition of the multiplication in $B \times G$, we see that $M'_s M'_t = \gamma[\phi(s,t)]M'_{st}$ for $s, t \in G$. The homomorphism of G into the projective unitary group corresponding to M' is of course just M. Thus a cocycle representative for the cohomology class of M is $\gamma[\phi]$. (This differs from the terminology of [6] to the extent of replacing σ by $1/\sigma$ for $\sigma \in Z^2(G,T)$.) Thus the cohomology class attached to M is just $\mathrm{tg}(\gamma)$ where tg is the transgression map of $H^1(B,T)$ onto $H^2(G,T)$.

We introduce the space $P(G)$ of classes of primary projective unitary representations of G, classified up to unitary equivalence, and its subspaces $P(G)^\alpha$ of classes with attached cohomology class α in $H^2(G,T)$. The correspondence of M with L established above preserves unitary equivalence, and we thus deduce a map ψ of $P(H)^0$ into $P(G)$.

THEOREM 2.6. *The map ψ is onto and $\psi(\tilde{L})$ is irreducible iff $\tilde{L} \in P(H)^0$ is irreducible. The counterimage of any point $\tilde{M} \in P(G)^\alpha$ consists of a collection $\tilde{L}_\gamma$ of elements, one for each $\gamma \in \mathrm{tg}^{-1}(\alpha)$ with the property that $\tilde{L}_\gamma$ restricts to a multiple of γ on B.*

Proof. These facts result immediately by applying Theorem 8.3 of [6] and the fact that the transgression map is surjective. Explicitly, if $\tilde{M}$ belongs to $P(G)^\alpha$, we may choose a Borel map M' of G into a unitary group so that M' represents $\tilde{M}$ and so that $M'_s M'_t = \gamma[\phi(s,t)]M'_{st}$, $s,t \in G$, where γ is some element of $H^1(B,T)$ with $\mathrm{tg}\gamma = \alpha$. Then we define $L_{(b,s)} = \gamma(b)M_s$, $b \in B$, $s \in G$. L is clearly a unitary representation of H and ψ maps its class into $\tilde{M}$.

The final statement of the theorem shows that ψ is biunique if and only if the transgression map tg is biunique. If this is the case we follow the terminology of Schur [9, p. 23], and call H a representation group for G. We shall examine more closely the structure of such representation groups in the following chapter. The following result of a more general nature is immediate from the explicit construction of splitting groups (Theorem 2.1 and Proposition 2.6).

PROPOSITION 2.7. *If G is almost connected, G has a representation group if and only if $H^2(G,T)$ is smooth.*

The results of [6] and Theorem 2.4 pose the following question. Let K be a closed normal subgroup of a locally compact group M, and denote by G the quotient M/K. Then it is shown in [6, Theorem 8.2], that there is a map f from $\hat{K}^G$ (classes of irreducible representation of K left fixed by the operation of G on K) into $H^2(G,T)$. The space $\hat{K}^G$ has a natural Borel structure [5]. If H is a splitting group for G, $H^2(G,T)$ receives a Borel structure as a quotient space of $H^1(B,T)$. (This Borel structure is not in general the Borel structure of the topology we have previously assigned to $H^2(G,T)$.) Is f a Borel function?

THEOREM 2.7. *In the above notation, let S be a Borel subset of $\hat{K}^G$ so that the Borel structure on S is analytic. Then f restricted to S is a Borel map into*

$H^2(G,T)$ where the latter space is given the Borel structure deduced from any splitting group.

Before proceeding with the proof, we first observe a simple consequence. If in this theorem, we take M to be a splitting extension of G by a central kernel K, then the Borel structure on $\hat{K}^G = \mathrm{Hom}(K,T)$ is just that deduced from the locally compact topology there. Since $\hat{K}^G$ is an analytic space and as it is easily verified that $-f$ is the transgression map, we easily deduce the following (cf. Theorem 2.2).

COROLLARY. *The Borel structure on $H^2(G,T)$ deduced from a splitting extension H of G is independent of the choice of the splitting group H.*

Proof of Theorem 2.7. Let H be a fixed splitting extension of G by a central kernel B, and let a be a cocycle in $Z^2(G,B)$ defining this extension. Let $\bar{a}$ be the inflation of a to M. Then viewing B as an M-module, we may form an extension $\bar{M}$ of M by B defined by the cocyle $\bar{a}$; we may represent $\bar{M}$ as pairs (b,m), $b \in B$, $m \in M$. It is easily verified from the nature of $\bar{a}$ that the set of elements of the form $(0,k)$ for $k \in K \subset M$ forms a normal subgroup canonically isomorphic to K. This subgroup is closed. (Observe that its coset space is a standard Borel space and apply Theorem 7.2 of [6].) We shall identify K with this subgroup since the Borel homomorphism $k \to (0,k)$ has closed range and thus embeds K topologically in $\bar{M}$. The quotient $\bar{M}/K$ is isomorphic to H, and also $\bar{K}$ is the direct sum of K and B, where $\bar{K}$ is the counterimage in $\bar{M}$ of K in M.

Let A be an irreducible representation of K whose equivalence class $\tilde{A}$ is invariant under G. Then A may be extended to M as an irreducible projective representation whose cohomology class is the inflation to M of $f(\tilde{A}) \in H^2(G,T)$. On the other hand, if γ is an element of B, then $A \times \gamma$ is an irreducible representation of $\bar{K} = K + B$ which is invariant under the operation of G on this group. It may therefore be extended to an irreducible projective representation C of $\bar{M}$. The construction of C [6, Theorem 8.2] shows immediately that the cohomology class attached to C is the inflation to $\bar{M}$ of $f(\tilde{A})/\mathrm{tg}(\gamma)$ where tg is the transgression homomorphism of $H^1(B,T)$ into $H^2(G,T)$ defined by the splitting extension H. It follows that $A \times \gamma$ extends to an irreducible representation of M if $f(\tilde{A}) = \mathrm{tg}(\gamma)$. Conversely let C be an irreducible representation of $\bar{M}$ restricting to an irreducible representation of $\bar{K}$; $(C)_{\bar{K}}$ is then of the form $A \times \gamma$ for $\tilde{A} \in K^G$, $\gamma \in \hat{B}$. By [6], if α denotes the inflation of $f(\tilde{A})/\mathrm{tg}(\gamma)$ to $\bar{M}$, there is an extension D of $A \times \gamma$ to $\bar{M}$ as a projective α representation, and an irreducible $\mathrm{tg}(\gamma)/f(\tilde{A})$ projective representation E of $G = \bar{M}/\bar{K}$ so that $C \simeq D \otimes \bar{E}$ where $\bar{E}$ is the inflation of E to M. If we restrict both sides of this equivalence to $\bar{K}$, it follows that $\bar{E}$ is one dimensional. We further conclude that $\mathrm{tg}(\gamma)/f(\tilde{A}) = 1$. Thus $f(\tilde{A}) = \mathrm{tg}(\gamma)$ is a necessary and sufficient condition that $A \times \gamma$ be the restriction to $\bar{K}$ an irreducible representation of $\bar{M}$.

If S is as in the statement of the theorem, we claim that the set X of $(\tilde{A},\gamma)$ in $S \times \hat{B}$ which are restrictions of irreducible representations of $\bar{M}$ is an analytic set. For if $\bar{M}^c$ and $\bar{K}^c$ denote the standard Borel spaces of concrete unitary representations of $\bar{M}$ and $\bar{K}$ [5], the operation r of restriction from $\bar{M}^c$ to $\bar{K}^c$ is evidently a Borel function. The set S^c of concrete irreducible representations corresponding to S is a Borel subset of K^c, and we immediately see that $S^c \times \hat{B}$ is a Borel subset of $\bar{K}^c$ since $\bar{K} = K + B$ and B is abelian. The set $r(\bar{M}^c) \cap (S^c \times \hat{B})$ is then an analytic set [4, §35]. Its projection into $S \times \hat{B}$, which is none other than X, is then also an analytic set since S is an analytic space.

Thus this subset X of $S \times \hat{B}$ consisting of those pairs $(\tilde{A},\gamma)$ which satisfy $f(\tilde{A}) = \mathrm{tg}(\gamma)$ is an analytic set. In the special case when tg is an isomorphism of B with $H^2(G,T)$, then the set X is just the graph of the map f restricted to S if B is identified to $H^2(G,T)$. It follows then from [4, §35, V, 2, p. 398] that f restricted to S is a Borel function. In the general case, an examination of the proof of the theorem just invoked shows that the argument may be used virtually verbatim to prove that f is a Borel function on S into the Borel structure on $H^2(G,T)$ obtained by viewing it as a quotient group of B. This completes the proof of the theorem.

Since it is possible to topologize $\hat{K}^G$ in a natural fashion [2], one may ask if the map f discussed above is also a continuous function into $H^2(G,T)$. We are unable to resolve this question.

Chapter III

1. We shall conclude by giving some examples and complements to the preceding.

If G is a compact group, the construction of Theorem 2.1 yields a collection of splitting groups for G and the circle group T. Since $H^2(G,R) = 0$, the only nonuniqueness involved is the choice of a cross section of $H^2(G,T)$ in $Z^2(G,T)$. This cross section is determined by the cocycle $\phi \in Z^2(G, \mathrm{Hom}(K,T))$ (see Theorem 2.1 for notation). The resulting splitting group, which is in fact a representation group in the terminology introduced above, we denote by H_ϕ.

Proposition 3.1. *Every representation group H of G is isomorphic to some H_ϕ as an extension of G by* $\mathrm{Hom}(H^2(G,T),T)$.

Proof. Let H be an extension of G by a central kernel B and let ψ be a cocycle defining the extension. The group $\mathrm{Hom}(B,T)$ is isomorphic to $H^2(G,T)$ and thus to the group K of Theorem 2.1. If $\beta \in \mathrm{Hom}(B,T)$, let $b_\beta(s,t) = \beta(\psi(s,t))$. Then since b_β is a cocycle representative of $\mathrm{tg}(\beta)$, the b_β form a cross section of $H^2(G,T)$ in $Z^2(G,T)$. We use this cross section in the construction of Theorem 2.1. The resulting cocycle $\phi \in Z^2(G,\mathrm{Hom}(K,T))$ of Theorem 2.1 is defined by $[\phi(s,t)]\beta = b_\beta(s,t) = \beta(\psi(s,t))$. It is clear now that when B is identified to

$\mathrm{Hom}(\mathrm{Hom}(B,T)) = \mathrm{Hom}(K,T)$ by duality that ϕ is identical with ψ. This completes the proof.

It is known that all H_ϕ need not be isomorphic when G is finite. We can prove the following which generalizes a result of Schur [9, Satz IV] for finite groups.

PROPOSITION 3.2. *All representation groups are isomorphic (as extensions of G by $B = \mathrm{Hom}(H^2(G,T),T)$ if $\mathrm{Hom}(G,T)$ is divisible (in particular if it is zero).*

Proof. Let H_ϕ and H_ψ be two representation groups for G. Since $H^1(G,T)$ is divisible, it is a direct summand in $C^1(G,T)$. Thus we may find an isomorphism f of $B^2(G,T)$ into $C^1(G,T)$ which inverts the differentiation operator δ_1. Now for fixed β in $H^2(G,T)$, $(\psi-\phi)\beta$ is an element of $B^2(G,T)$. Let us denote by $\gamma(\beta)$ the element $f((\psi-\phi)\beta)$ in $C^1(G,T)$. Since f is additive the map $\beta \to [\gamma(s)](\beta)$ for fixed s in G is a homomorphism of $H^2(G,T)$ into T. It is of course trivially continuous as $H^2(G,T)$ is discrete; moreover, the argument given in the proof of Theorem 2.1 may be repeated almost unchanged to show that $\gamma(s)$ is a Borel function from G into $B = \mathrm{Hom}(H^2(G,T),T)$. Finally we calculate that $\psi-\phi=\delta_1\gamma$ as elements of $Z^2(G,B)$. This completes the proof of the proposition since then ψ and ϕ define the same element of $H^2(G,B)$.

It follows from [8] that $H^2(G,T)$ is a discrete torsion group, and by duality B is a compact totally disconnected group. Any representation group H_ϕ is then compact and resembles a "covering group" of G since the kernel of the projection onto G is zero dimensional. The following shows that H_ϕ behaves in another respect like a covering group.

PROPOSITION 3.3. *If G is connected then any H_ϕ is also connected.*

Proof. Let H' be the connected component of H_ϕ. If H' is proper, we may find an open and closed proper normal subgroup H of H_ϕ containing H' (since H_ϕ/H' is compact totally disconnected). Let $K = H \cap B$ where B is as usual the kernel of the projection onto G. Since the projection of H onto G is an open subgroup, it is all of G, and it follows that $H_\phi = BH$. It is immediate now that

$$H_\phi/K \simeq B/K + H/K$$

(direct sum) topologically and algebraically. If γ is an element of $\mathrm{Hom}(B,T)$ vanishing on K, then γ viewed as a character of B/K extends to H_ϕ/K, and thus γ as character of B extends to H_ϕ. It follows then that $\mathrm{tg}(\gamma) \in H^2(G,T)$ is zero where tg is the transgression map defined by the splitting group H_ϕ. Since tg is an isomorphism, γ must be the trivial character and by duality $K = B$. Thus $H \supset B$ and since $H_\phi = BH$, $H = H_\phi$. But this contradicts the choice of H'.

If G is a compact connected Lie group, it is easy to say what all possible representation groups are using Proposition 2.1 of [8]. If $\tilde{G}$ is the universal covering group of G, then $G \simeq G/\pi_1(G)$. Let F be a subgroup of $\pi_1(G)$ complementary to the torsion subgroup of $\pi_1(G)$. Then we see that $H = G/F$ is a representation group, and every representation of G is of this form for suitable F. Bargmann [1, Theorem 7.3] has previously proved a result very much akin to this. This offers an example of Proposition 3.2 for if $H^1(G,T)$ is divisible it is zero and G is semi-simple. The only representation group is $\tilde{G}$. If G is the group $T + R_3$ where R_3 is the 3 dimensional rotation group, then G has two nonisomorphic representation groups; namely, $U(2,C)$ and $T + SU(2,C)$.

It was remarked in [8] that any countable torsion group A could appear as $H^2(G,T)$ for suitable choice of G. It is easy to construct such a G. The given group A may be represented as a quotient C/D where C is a possible infinite sum $\sum Z/n_iZ$ of cyclic groups. Let $H = \prod SU(n_i, C)$. Then the center E of H is $\prod Z/n_iZ$, which is isomorphic to the dual of C. Let B be the closed subgroup of E corresponding to those characters of C which vanish on D, and let $G = H/B$. It follows from Proposition 2.1 and Theorem 2.1 of [8] that $H^2(H,T) = 0$, and it is clear that $H^1(H,T) = 0$. The restriction-inflation sequence for the covering H of G then implies that the transgression map of $H^1(B,T)$ into $H^2(G,T)$ is bijective. But $H^1(B,T)$ is isomorphic to A by duality and we are finished. This also shows that H is (the) representation group of G.

2. None of the foregoing explicitly includes the case when G is an (infinite) discrete group. A somewhat simpler and more direct method is, however, applicable in this case. If G is discrete and A is a compact G-module, then the groups of normalized cochains $C^n(G,A)$ are compact in the product topology, and the differentiations are continuous. It follows that $Z^n(G,A)$ and $B^n(G,A)$ are closed subgroups and we assign to $H^n(G,A)$ the resulting compact Hausdorff quotient topology. The homomorphisms in the exact sequence of cohomology corresponding to a short exact sequence $0 \to A' \to A \to A'' \to 0$ of G modules are continuous as one easily sees.

We shall show that there is a splitting group for G and A at least when A is a torus. By the duality theory for compact groups we observe that those compact groups which are (possibly infinite) products of circles are just the injectives in the category of compact abelian groups. Thus any product of injectives is injective and any quotient of an injective is an injective. If A is a torus, it follows that $C^n(G,A)$ and $B^n(G,A)$ are injective for all n. In particular $B^2(G,A)$ is a topological and algebraic summand in $Z^2(G,A)$. Thus we can find a continuous homomorphism $\beta \to b_\beta$ of $H^2(G,A)$ into $Z^2(G,A)$ so that $b_\beta \in \beta$. As before (Theorem 2.1) we let $[\phi(s,t)]\beta = b_\beta(s,t)$ for $s,t \in G$. Then it is trivial that ϕ belongs to $Z^2(G,B)$ where $B = \mathrm{Hom}(H^2(G,A),A)$ is given the compact open (discrete) topology. The cocycle ϕ defines an extension H_ϕ of G by the G kernel B.

THEOREM 3.1. (1) *In the above notation, H_ϕ is discrete, and is a splitting group for G and A. The splitting group topology coincides with the natural compact Hausdorff topology on $H^2(G,A)$.*

(2) *If A is the circle group T with trivial operation, H_ϕ is a representation group and any representation group is isomorphic to some H_ϕ. All representation groups are isomorphic if $G/[G,G]$ is free abelian.*

Proof. The proofs are carried out exactly as in the previous section. For the final statement the analogy with Proposition 3.2 shows that it suffices to know that $Z^1(G,T)$ is a topological and algebraic direct summand in $C^1(G,T)$. But the condition of the theorem is equivalent to the fact that $Z_1(G,T)$ is injective as compact abelian group.

The final statement of the theorem generalizes in another direction the result of Schur [9] for finite groups quoted above.

3. We consider finally some examples of connected locally compact groups. Iwasawa [3] has shown that in such a group G we may find a normal series $G = G_0 \supset G_1 \supset G_2 \supset N \supset 0$ where G_0/G_1 is a semi-simple Lie group, G_1/G_2 is a compact abelian and totally disconnected, G_2/N is compact and $\mathrm{Hom}(G_2/N, T) = 0$. Finally N is the radical (maximal normal connected solvable subgroup) of G. The subgroup G_1 is of course not necessarily unique, but for our purposes this nonuniqueness will not matter.

In what follows the circle group T will be regarded as a module with trivial operators for each of the several groups H considered. The following is a restatement of well-known facts.

PROPOSITION 3.4. *If H is a semi-simple connected Lie group, its universal covering group $\tilde{H}$ is a representation group for H.*

In particular, $H^2(G/G_1,T)$ is isomorphic to $\mathrm{Hom}(\pi_1(G/G_1),T)$ topologically and $H^2(G/G_1,T)$ is smooth.

We consider the extension G/G_2 of G/G_1 by G_1/G_2. Let the line R be viewed as the covering group of T and as a trivial G-module. By Proposition 1.2, the inflation map of $H^2(G/G_1,R)$ into $H^2(G/G_2,R)$ is an isomorphism onto. Also the projection of $H^2(G/G_2,R)$ into $H^2(G/G_2,T)$ is a subgroup of finite index, consisting of those homomorphisms of $\pi_1(G/G_1)$ into T which vanish on the torsion part of $\pi_1(G/G_1)$. Using Theorem 2.3 and Proposition 2.6 as a criterion of smoothness, we see that $H^2(G/G_2,T)$ is smooth if and only if the kernel of the inflation map of $H^2(G/G_1,T)$ into $H^2(G/G_2,T)$ is closed. This kernel is the image under the transgression homomorphism tg of $H^1(G_1/G_2,T)$. It is clear that $H^1(G/G_2,T)$ is the zero group and so the transgression map is 1-1. Since $H^1(G/G_2,T)$ is discrete and tg is continuous (Theorem 2.4), the kernel of the inflation is closed if and only if $H^1(G_1/G_2,T)$ is finite. We immediately deduce the following.

PROPOSITION 3.5. *The group $H^2(G/G_2,T)$ is smooth if and only if G/G_2 is a Lie group.*

The following example shows what can happen at the opposite extreme. Let G be $SL(2, R)$; the fundamental group is isomorphic to Z. Let ϕ be a cocycle of G with values in $\pi_1(G)$ which defines the universal covering group $\tilde{G}$ as an extension of G by $\pi_1(G)$. Let Z^* denote the compactification of Z in the topology defined by using all ideals as a basis of neighborhoods at 0. Let ϕ^* be the corresponding 2 cocycle of G with values in Z^*, and let H be the extension of G by Z^* so defined. We contend that the inflation map of $H^2(G,T)$ into $H^2(H,T)$ is surjective. For this, let E_r^{ji} be the spectral sequence of the group extension H of G by Z^* abutting to $H^*(H,T)$. It suffices to show that $E_2^{11} = E_2^{02} = 0$. Since $H^1(Z^*,T)$ is discrete and G is connected, $E_2^{11} = 0$ by Theorem 1.1 of [8]. Now Z^* may be represented as the projective limit $\lim p\cdot_n(Z/n!Z)$ of cyclic groups. That $H^2(Z/n!Z,T) = 0$ is a well-known fact; it then follows from Theorem 2.1 of [8] that $H^2(Z^*,T) = E_2^{02} = 0$.

As we have remarked above, the transgression map of $H^1(Z^*,T)$ into $H^2(G,T)$ is 1-1. Now the latter group is topologically and algebraically isomorphic to $H^2(Z,T) = T$, and we see without difficulty that $H^2(H,T)$ is topologically and algebraically isomorphic to the quotient $H^1(Z,T)/H^1(Z^*,T)$. This quotient is just $T/(Q/Z)$ where Q denotes the rational numbers, or equivalently isomorphic to R/Q. The splitting group topology on $H^2(H,T)$ is then completely trivial (two open sets).

We continue with the discussion of the normal series of Iwasawa in G. There is no difficulty in passing from G/G_2 to G/N.

Proposition 3.6. *The inflation map from $H^2(G/G_2,T)$ into $H^2(G/N,T)$ is* 1-1 *and has denumerable cokernel. Consequently $H^2(G/N,T)$ is smooth if and only if $H^2(G/G_2,T)$ is.*

The proof follows by exactly the same method as used for Proposition 3.5, using now the fact that $H^1(G_2/N,T) = 0$.

We cannot proceed further down the normal series into the radical N of G unless some special assumptions are made. In particular if N is compact, the same argument as above shows that the inflation map from $H^2(G/N,T)$ into $H^2(G,T)$ is surjective. This follows since $H^2(N,T) = 0$ in virtue of Theorem 2.1 of [8], and the fact that $H^1(G/N,H^1(N,T)) = 0$ since $H^1(N,T)$ is discrete and G/N is connected. It remains to evaluate the range of the transgression homomorphism of $H^1(N,T)$ into $H^2(G,T)$. This becomes particularly elegant if N is a finite or infinite product $\prod_1^n T$ of circles. The extension of G/N by N is central and so G is characterized by a family of n elements $\{\alpha_i\}^n$ from $H^2(G,T)$, $n = 1,\cdots,\infty$. It is immediate that the range of the transgression is just the group algebraically generated by the $\{\alpha_i\}$. This determines $H^2(G,T)$ directly in terms of $H^2(G/N,T)$ and the cohomology class of the extension of G/N by N.

Let G be a connected group with compact radical, V a finite dimensional vector G-module, and H an extension of G by the G-module V. We give a necessary and sufficient condition that $H^2(H,T)$ be smooth for this class of locally compact groups. Let G_1 be the subgroup discussed above in the normal series in G. It is known that $H^1(G/G_1, W) = 0$ and that $H^2(G/G_1, W)$ is isomorphic to $\mathrm{Hom}(\pi, W^{G/G_1})$ where $\pi = \pi_1(G/G_1)$ for any finite dimensional G/G_1-module W. Since G_1 is compact, these facts combined with Proposition 1.2 show that $H^1(G, W) = 0$ and $H^2(G, W) = \mathrm{Hom}(\pi, W^G)$. The vanishing of the one dimensional cohomology implies that any finite dimensional G-module is completely reducible. Returning to our group H, we now denote by ϕ the element of $\mathrm{Hom}(\pi, W^G)$ which describes this extension of G by V.

The argument given previously analyzing $H^2(G, T)$ shows that the closure L of the identity element in the splitting group topology is contained within the image of the inflation map of $H^2(G/G_1, T) = \mathrm{Hom}(\pi, T)$ in $H^2(G, T)$. The counterimage of L in $\mathrm{Hom}(\pi, T)$ is a closed subgroup A of this group. Let B denote the subgroup of π consisting of the common zeroes of A. The criterion for smoothness can be stated as follows in terms of B and ϕ alone.

PROPOSITION 3.7. *The cohomology group $H^2(H, T)$ is smooth if and only if the kernel of ϕ is contained in B and the range of ϕ in V^G is closed.*

The proof can be carried out easily using Proposition 2.6 as a criterion for smoothness. The complete reducibility of finite dimensional representations of G is needed. Roughly, the first condition says that any nonsmoothness in $H^2(G, T)$ is factored out in the process of inflation to H, and the second condition says that no new nonsmoothness arises in the process of inflation. Observe that when $H^2(G, T)$ is already smooth, B is all of π, and the first condition becomes vacuous. We shall omit the details.

If G is solvable or a nilpotent Lie group, $H^2(G, T)$ can be rough or smooth almost at will. A central extension G of $R + R$ by $T + T$ is defined by a pair of elements α and β in $H^2(R + R, T)$. It is easy to see that the inflation map from $H^2(R + R, T)$ to $H^2(G, T)$ is surjective and that its kernel is generated by α and β. Since $H^2(R + R, T)$ is isomorphic to R, it is easily seen that $H^2(G, T)$ is not smooth if and only if α/β is irrational (where α and β are now viewed as real numbers).

Our final example shows that a locally compact group may fail to have a splitting group. In fact if G is the reals plus an infinite direct sum of the integers $R + \sum_{i=1}^{\infty} Z_i$, then by [6, p. 305],

$$H^2(G, T) \simeq H^2(R, T) + H^2\left(\sum_{i=1}^{\infty} Z_i, T\right) + \mathrm{Hom}\left(\sum_{i=1}^{\infty} Z_i, \mathrm{Hom}(R, T)\right).$$

The first summand is zero, and the final one is isomorphic to $\prod_{i=1}^{\infty} R_i$. If G has splitting group, then the inclusion of the subgroup $R + Z_i$ (the copy of Z is in the ith place) induces a continuous homomorphism (Theorem 2.3) of $H^2(G, T)$ into $H^2(R + Z_i, T)$. The latter group is just $\mathrm{Hom}(Z_i, \mathrm{Hom}(R, T)) = R_i$ since $H^2(R, T) = H^2(Z, T) = 0$. The map of $H^2(G, T)$ into R_i maps $H(\sum_{i=1}^{\infty} Z_i, T)$ into zero and $\prod_{i=1}^{\infty} R_i$ into its ith component. It follows that $H^2(G,T)/H^2(\sum_{i=1}^{\infty} Z_i,T)$ is mapped continuously and biuniquely into $\prod_{i=1}^{\infty} R_i$, where the former group has a separable locally compact topology deduced from the splitting group topology, and the latter has the product topology. By the closed graph theorem, the homomorphism is bicontinuous. But this is glaring nonsense for then $\prod_{i=1}^{\infty} R_i$ would be locally compact in the product topology.

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